

Adaptable nonlinear bimode metamaterials using rigid bars, pivots, and actuators.

Graeme Walter Milton

Department of Mathematics, University of Utah, Salt Lake City UT 84112, USA

Abstract

A large family of periodic planar non-linear bimode metamaterials are constructed from rigid bars and pivots. They are affine materials in the sense that their macroscopic deformations are only affine deformations: at large distances any deformation must be close to an affine deformation. Bimode means that the paths of all possible deformations of Bravais lattices that preserve the periodicity of the lattice lie on a two dimensional surface in the three dimensional space of invariants describing the deformation (excluding translations and rotations). By adding two actuators inside a single microscopic cell one can control the macroscopic deformation, which may be useful for the design of adaptive structures. The design of adaptable nonlinear affine trimode metamaterials (for which the macroscopic deformations lie within a three-dimensional region in the space of invariants) is discussed, although their realization remains an open problem. Examples are given of non-affine unimode and non-affine bimode materials. In these materials the deformation can vary from cell to cell in such a way that the macroscopic deformation is non-affine.

Keywords: A. microstructures, B. inhomogeneous material; B rods and cables; B foam material; bimode material

1 Introduction

Here we consider the deformation of planar periodic lattices of rigid (inextensible) bars with pivot joints. Since they are periodic they have a two dimensional Bravais lattice consisting of points

$$\mathbf{x} = i\mathbf{u} + j\mathbf{v}, \quad (1.1)$$

as i and j range over all integers, where \mathbf{u} and \mathbf{v} are the lattice vectors. Let \mathbf{F} be the deformation matrix with the vectors \mathbf{u} and \mathbf{v} as columns. Then under a rotation \mathbf{R} , since \mathbf{u} and \mathbf{v} transform to $\mathbf{R}\mathbf{u}$ and $\mathbf{R}\mathbf{v}$ it follows that \mathbf{F} transforms to $\mathbf{R}\mathbf{F}$ leaving the symmetric Cauchy-Green matrix $\mathbf{A} = \mathbf{F}^T\mathbf{F}$ invariant. There are of course many other Bravais lattices. In particular, there are lattices with lattice vectors

$$\mathbf{u}' = k\mathbf{u} + \ell\mathbf{v}, \quad \mathbf{v}' = m\mathbf{u} + n\mathbf{v}, \quad (1.2)$$

for any choice of the integers k , ℓ , m and n such that \mathbf{u}' and \mathbf{v}' are independent. The corresponding deformation matrix \mathbf{F}' , which has \mathbf{u}' and \mathbf{v}' as columns, is given by

$$\mathbf{F}' = \mathbf{F}\mathbf{M}, \quad \text{where } \mathbf{M} = \begin{pmatrix} k & m \\ \ell & n \end{pmatrix}, \quad (1.3)$$

and \mathbf{M} is non-singular. Other Bravais lattices will correspond to matrices \mathbf{M} with elements that are not necessarily integers. As the material continuously deforms, with the Bravais lattice having lattice vectors \mathbf{u} and \mathbf{v} undergoing an affine transformation, \mathbf{F} follows some trajectory $\mathbf{F}(t)$ beginning at $t = t_0$. Associated with the deformation is a path

$$\mathbf{C}(t) = [\mathbf{F}(t_0)^T]^{-1} \mathbf{F}(t)^T \mathbf{F}(t) [\mathbf{F}(t_0)]^{-1} \quad (1.4)$$

in the three dimensional space of symmetric matrices beginning at $\mathbf{C}(t_0) = \mathbf{I}$. If we chose a different Bravais, such as that in (1.2), this path, for the same deformation, would be the same since

$$[\mathbf{M}^T \mathbf{F}(t_0)^T]^{-1} \mathbf{M}^T \mathbf{F}(t)^T \mathbf{F}(t) \mathbf{M} [\mathbf{F}(t_0) \mathbf{M}]^{-1} = [\mathbf{F}(t_0)^T]^{-1} \mathbf{F}(t)^T \mathbf{F}(t) [\mathbf{F}(t_0)]^{-1}. \quad (1.5)$$

On the other hand there can be continuous deformations in which the larger lattice undergoes an affine transformation, but the smaller lattice is distorted. So it is possible that there exist additional paths of deformation $\mathbf{C}(t)$ for the larger lattice which are not accessible to the smaller lattice, see figure 1 and the examples in Kapko et al. (2009), for instance.

A material is classed as null-mode if the only possible continuous deformation path for any Bravais lattice is the point $\mathbf{C}(t) = \mathbf{I}$. Some internal motions could be possible but macroscopically the material is rigid to periodic deformations (in the sense that for any Bravais lattice the only continuous deformations that keep the periodicity of that lattice only rotate and translate the lattice). A material is classed as unimode if it is not null-mode and the possible deformations $\mathbf{C}(t)$ for any Bravais lattice all lie on the same one dimensional curve. A material is classed as bimode if it is not null-mode or unimode and the possible deformations $\mathbf{C}(t)$ for any Bravais lattice all lie on the same two dimensional surface. A material is classed as trimode if it is not null-mode, unimode, or bimode. These definitions extend the definitions of unimode, bimode and trimode introduced by Milton and Cherkaev (1995) in the context of planar linear elasticity (see also chapter 30 of Milton 2002). The definitions extend to three dimensions in the obvious way, and in three dimensions one can also have quadramode, pentamode, and hexamode nonlinear materials (Milton 2012). Recently, a class of linearly elastic pentamodes suggested by Milton and Cherkaev (1995) were experimentally realized by Kadic et al. (2012).

Periodic materials made from bars and pivots can be further classed into “affine materials” for which an affine transformation is the only possible macroscopic deformation, and non-affine materials like that in figure 1, which can have other macroscopic deformations. In these non-affine materials the deformation varies from cell to cell in such a way that the macroscopic deformation is non-affine. More precisely a material is affine if and only if any deformation $\mathbf{x}'(\mathbf{x})$ (defined only for points \mathbf{x} on the rigid bars which get moved to $\mathbf{x}'(\mathbf{x})$) necessarily approaches $\mathbf{B}\mathbf{x} + \mathbf{b}$ as $|\mathbf{x}| \rightarrow \infty$, for some non-singular matrix \mathbf{B} and vector \mathbf{b} (dependent on the deformation) to within terms of $o(|\mathbf{x}|)$. A material which isn’t affine is non-affine. A material is macroscopically rigid if and only

if any deformation $\mathbf{x}'(\mathbf{x})$ necessarily approaches $\mathbf{R}\mathbf{x} + \mathbf{b}$ as $|\mathbf{x}| \rightarrow \infty$, for some rotation \mathbf{R} and vector \mathbf{b} , to within terms of $o(|\mathbf{x}|)$. We do not know if a null-mode material is necessarily macroscopically rigid.

There are lots of examples of affine unimode materials, see the companion paper (Milton 2012) and references therein. Section 2 of this paper gives an example of a non-affine unimode material and an example of a non-affine bimode material. It also shows in section 3 how a large family of affine bimode materials can be constructed using bars and pivots. [Based on the figures of Kapko et al. (2009) the Roman mosaic net may be another example of an affine bimode material, but we are not sure of this.] It is easy to construct non-affine trimode materials from bars and pivots: figure 1 gives an example. We do not have any examples of affine trimode materials, although in section 4 we suggest a possible route towards obtaining them.

Although the constructions are two-dimensional they can easily be extended into the third dimension, to obtain three-dimensional materials, by replacing rigid bars with rigid sheets, and pivot joints by edge joints. Then these structures can then be replaced by trusses of rigid bars and pivots. In this transformation two dimensional n -mode materials become three dimensional n -mode materials, and affineness, or non-affineness, is preserved.

Guest and Hutchinson (2003) raised the interesting question as to whether it is possible to build a periodic pin-jointed structure of rigid bars having no easy modes of deformation and such that replacing any bar by an actuator and changing its length leads to a change of the geometry of the structure, and not to self stress. They pointed out this is relevant to designing adaptive structures. They found that no such periodic pin-jointed structures of rigid bars can exist. On the other hand, the affine bimode materials considered here have the property that they become macroscopically rigid if two (appropriately placed) bars are added to just one cell in the structure. Changing the lengths of these bars (within limits) leads only to a change of the geometry of the structure, and not to self stress. Replacing the bars by actuators leads to interesting adaptive structures.

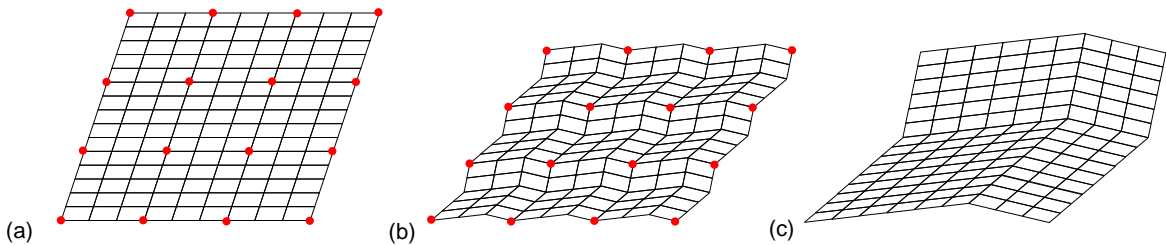


Figure 1: A parallelogram array of rigid bars can deform from (a) to (b). The red dots denote the Bravais lattice which undergoes an affine transformation. As a result the array is not unimode, but instead trimode. It is a non-affine material as it has macroscopic deformations which are non-affine as in (c).

2 Non-affine unimode and bimode materials

To construct a non-affine unimode material we construct a rectangular box which remains rectangular as the microstructure inside the box deforms, with two opposing sides remaining rigid. The key to the construction is Hart’s A frame (Hart 1877) which is a device for obtaining exact straight line motion: see figure 2(a). The mechanism has the property that the lengths AP and BP remain equal as the structure deforms. Thus the point P traces out a vertical straight line which is a segment of the perpendicular bisector of AB . As a consequence if we add supports as in figure 2(b) we obtain a table which deforms as an isosceles trapezoid $GHJK$. Attaching two such tables, which are 180° rotations of each other, by their feet, as in figure 2(c), gives the unit cell of a unimode material such that λ_2 is free to vary (in an interval) but λ_1 remains fixed.

To construct a non-affine bimode material we construct a rectangular box which remains rectangular as the microstructure inside the box deforms and is such that the two side lengths are free to vary independently within limits. This is done by adding an additional table (with “the feet extending beyond the table top”), and another table which is a 180° rotation of it and placing them back to back with the structure of figure 2(c) to give the unit cell structure of figure 3.

That these unimode and bimode structures are non-affine is easily seen from figure 4.

3 Bimode materials with structure controlled by two actuators

The first key component in the design of the desired bimode structures is what we call a u-structure (u for unimode), which is a triangle ABC with one internal degree of freedom. An example is shown in figure 5. A u-structure has three support points labeled A , B and C , and is such that the lengths AB , BC and CA are not rigid but are positive valued continuous functions $f(t)$, $g(t)$, and $h(t)$ of some variable t which varies in some range $t_0^- \leq t \leq t_0^+$ as the structure deforms. Thus specification of one length determines the other two lengths if not uniquely, then within a finite (usually small) number of possibilities. The u-structure could be a triangle of unimode metamaterial, and from the results of Milton (2012) it follows that $(f(t), g(t), h(t))$ can be made arbitrarily close to any desired trajectory in three-dimensional space. It is helpful to assume the structure remains inside the triangle ABC for a some interval of values of t . Deformations with t outside this interval may also be considered but then one runs the risk of the structure colliding with other structures.

The second key component in the design of the desired bimode structures is what we call a b-structure (b for bimode). It is a quadrilateral $ABCD$ with two internal degrees of freedom. It has sides with lengths AB , BC , CD and DA that are not rigid, but such that the lengths of any pair of adjoining sides, such as DA and AB can range anywhere inside some open connected region of two-dimensional space (i.e. they are not confined to lie on a curve) and that knowledge of these two lengths completely specifies the quadrilateral up to a finite number of possibilities. In other words given the lengths DA and AB (or any other pair of adjoining lengths) one can determine (up to a finite number of possibilities) the included angle DAB and the lengths of the two remaining

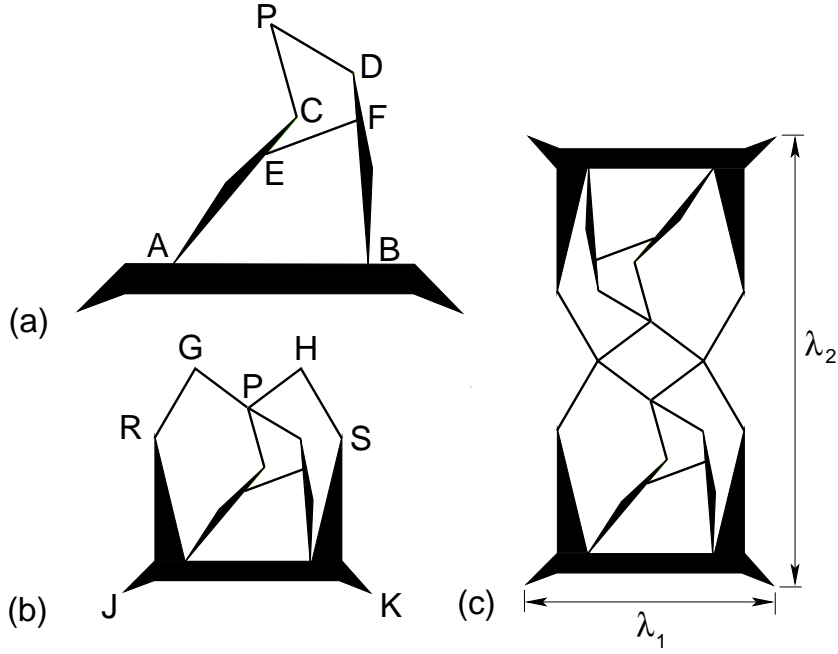


Figure 2: Figure (a) shows Hart's A frame. The black regions are rigid material. The lengths are such that $AC = BD = AB = a$, $PC = PD = EF = b$, and $CE = DF = c$ and $b^2 = ac$. For example, one may take $a = 4$, $b = 2$ and $c = 1$. The triangle APB remains isosceles as the structure deforms. In (b) is a (inverted) "table": we add two triangular pillars to the base of (a) and two bipod supports with lengths $RG = SH$ and $GP = PH$ so that $GHJK$ remains an isosceles trapezoid as the structure deforms, and JK remains rigid. If we attach a table to an inverted table, with the same structure but rotated by 180° as in (c) we obtain the unit cell of the desired non-affine unimode material: λ_2 is free to vary (in an interval) but λ_1 remains fixed.

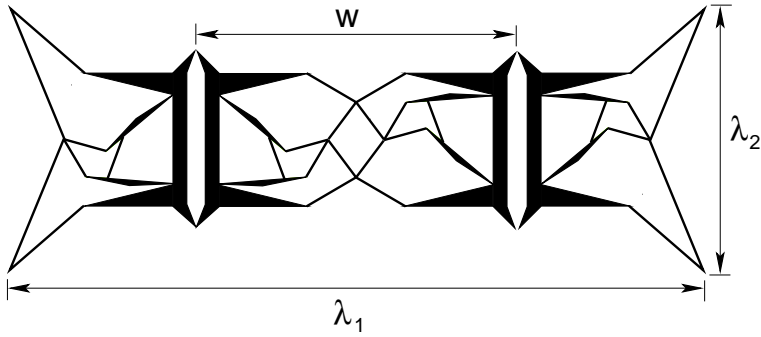


Figure 3: By combining four tables as shown here, we obtain the unit cell of the desired non-affine bimode material. Note that we are free to adjust λ_2 (within limits) and given λ_2 we are still free to adjust λ_1 by varying w .

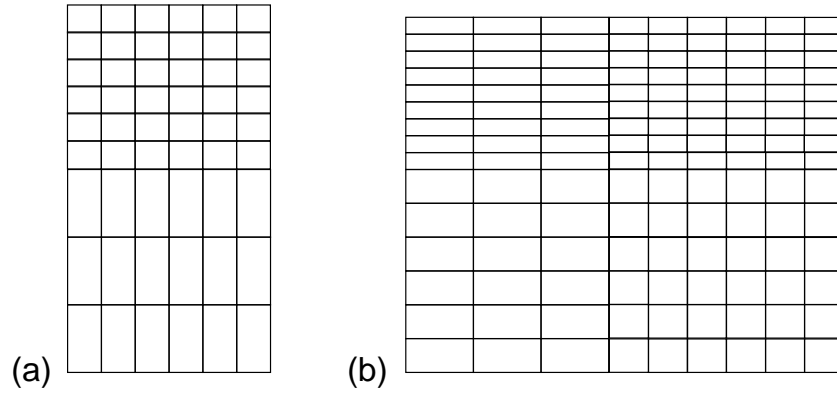


Figure 4: Figure (a) shows an exaggerated macroscopic deformation of a unimode material with each cell having the structure of figure 2. Figure (b) shows an exaggerated macroscopic deformation of a bimode material with each cell having the structure of figure 3. Therefore both are non-affine materials.

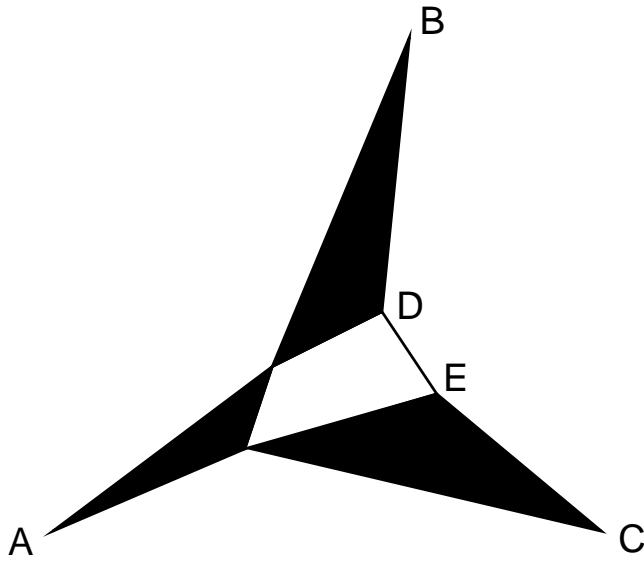


Figure 5: Example of a u-structure. The black triangles are rigid and DE is a rigid bar.

sides BC and CD . Additionally, the quadrilateral does not remain a trapezoid for arbitrary deformations of the structure, or if it does the lengths of the two parallel sides are not slaves to each other but instead determine the shape of the trapezoid.

Note that the quadrilateral of figure 6(a) is not a b-structure even though it has two internal degrees of freedom. The pair of adjoining lengths AB and BC are confined to a curve and do not determine the other lengths CD and DA . There are continuous deformations of the structure, such as from figure 6(a) to figure 6(b) that leave the triangle ABC unchanged, but not the quadrilateral $ABCD$. As a consequence, if we tile all space with these quadrilaterals and 180° rotations of them, then the resulting tiling can deform in many ways, such as illustrated in 6(c): the resulting structure is a non-affine trimode material and has many internal degrees of freedom.

Neither is the trapezoid of figure 7 a b-structure when the length AB always remains twice (or some constant times) the length CD , as the structure deforms. The structure is assumed to have two internal degrees of freedom, one being the length CD . So even with the length CD held constant the structure can deform from (a) to (b). A tiling of space by such trapezoids can deform in many ways as seen from figure 7(c)

On the other hand, the structure of figure 8(a) is a b-structure. When it and its 180° rotation are tiled to fill all space it looks like the bimode material of figure 8(b). This bimode material, and similar bimode materials formed from other b-structures, have only two internal degrees of freedom. To see this, consider the quadrilateral of 9(a), representing the structure of figure 8(a), tiled to fill all space. Now consider two adjoining tiles, as in 9(b), and suppose two actuators are added to cell 1, along say the lines BC and AD , and one actuator is added to cell 2, along say the line EF slaved so that tile 2 is always a 180° rotation of tile 1, even as tile 1 deforms. No other actuators are added to the structure. If tiles 3 and 4 have the same shape and orientation as tiles 1 and 2 then they fit together perfectly as in figures 9(c) and 9(d). Suppose however tile 3 deforms as in figure 9(e). This changes the length DG and the angle CDG . Tile 4 has to deform so the length CD remains unchanged and the length DH changes to the new value of the length DG . On the other hand, because tile 4 contains a b-structure the included angle CDH will be determined (within a finite number of possibilities) by the lengths CD and DH , and will not in general equal the value of the new angle CDG : there will be an angle mismatch as in figure 9(f). To avoid this mismatch the tiles 3 and 4 must deform as in 9(d). Similar considerations apply if we consider the deformation of the cells 5 and 6 and the cells 7 and 8 adjoining this structure as in figure 9(g), and by induction for the deformation of the entire lattice of cells. Thus the deformation of the entire lattice is controlled by the two actuators in cell 1 and the one actuator in cell 2 which is slaved. In fact we do not even need the slave actuator in cell 2. If cell 2 deforms but cell 1 does not, then the deformation of cell pairs 3 and 4, 5 and 6, and 7 and 8 in figure 9(g) will be slaved to it, and in general there will be a misfit when we come to cell 9.

This b-structure with its two internal actuators is an example of what we call a bimode transformer material: its macroscopic deformation is controlled by the two actuators in the material, and lies along a two dimensional surface in the three dimensional space of invariants describing the macroscopic deformation (excluding translations and rotations). From a practical viewpoint it would be better to have a sparse array of slaved actuators rather than just two, to avoid enormous stresses on the actuators when the structure is loaded. Also if the bars are not completely rigid, the desired deformation would not be

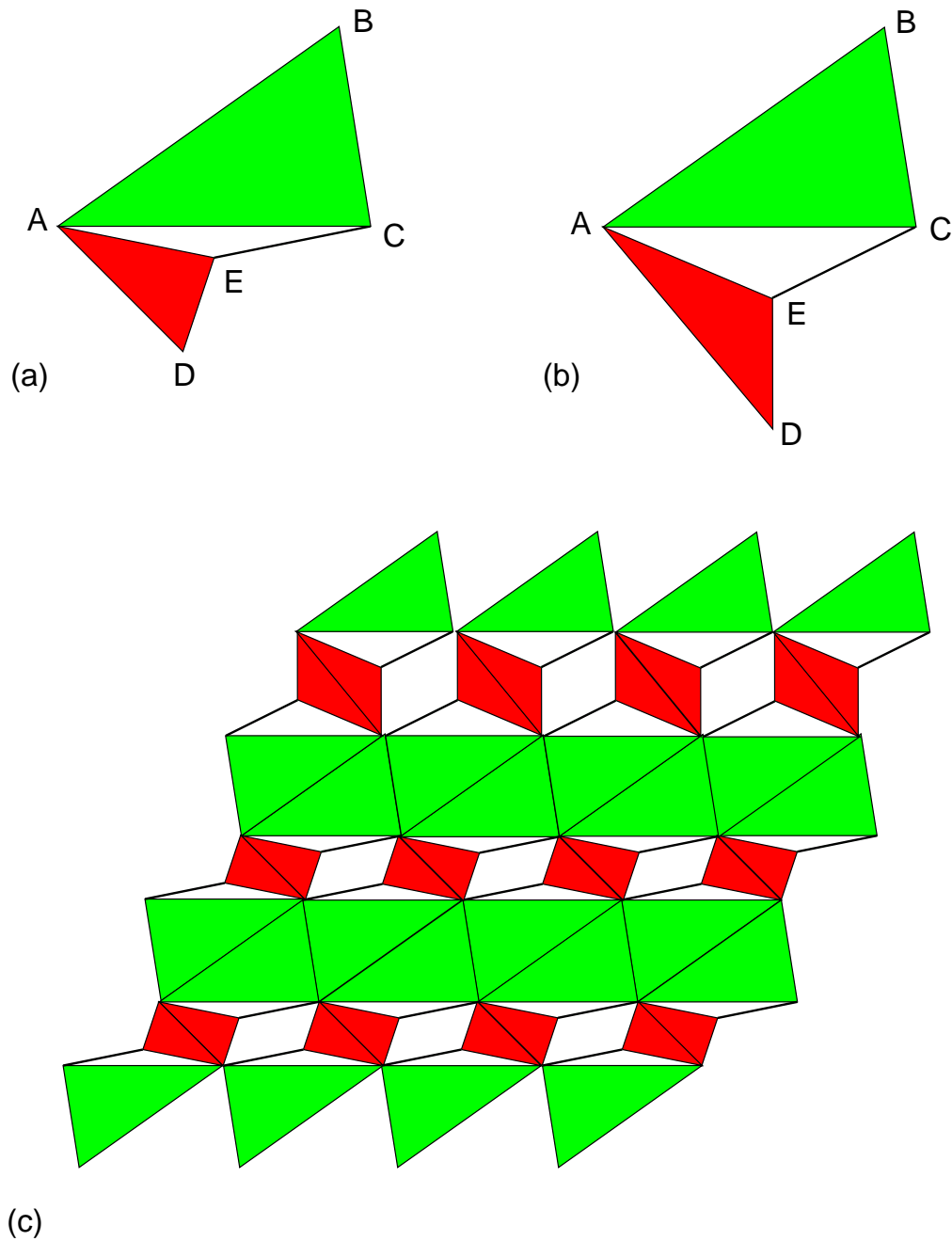


Figure 6: Figure (a) shows a quadrilateral $ABCD$ which is not a b-structure, but can deform as in (b) leaving the triangle ABC unchanged. As a result the tiling of space by these quadrilaterals can deform in many ways as illustrated, for example, in (c). Here the green and red triangles are u-structures and CE is a bar of fixed length

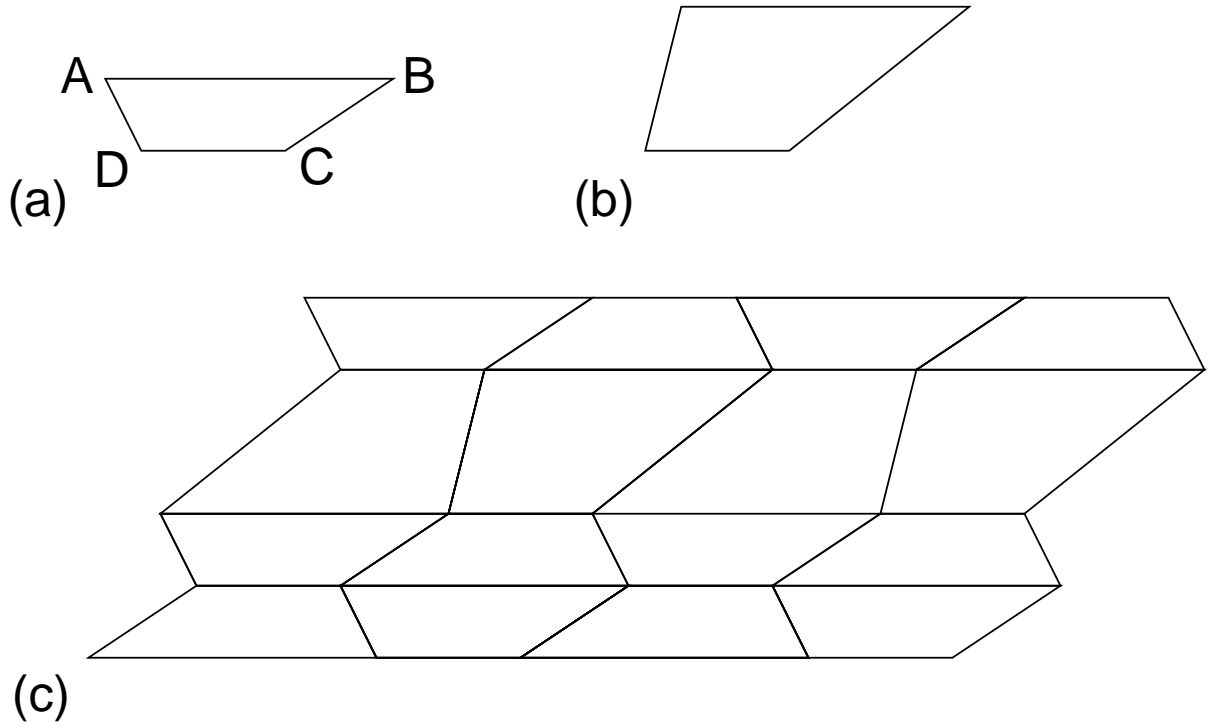


Figure 7: Figure (a) shows a trapezoid $ABCD$ which is not a b-structure when the length AB always remains twice (or some constant times) the length CD . If the length of CD remains fixed the structure can deform from (a) to (b), and a lattice of such cells can deform in many ways as illustrated, for example, in (c).

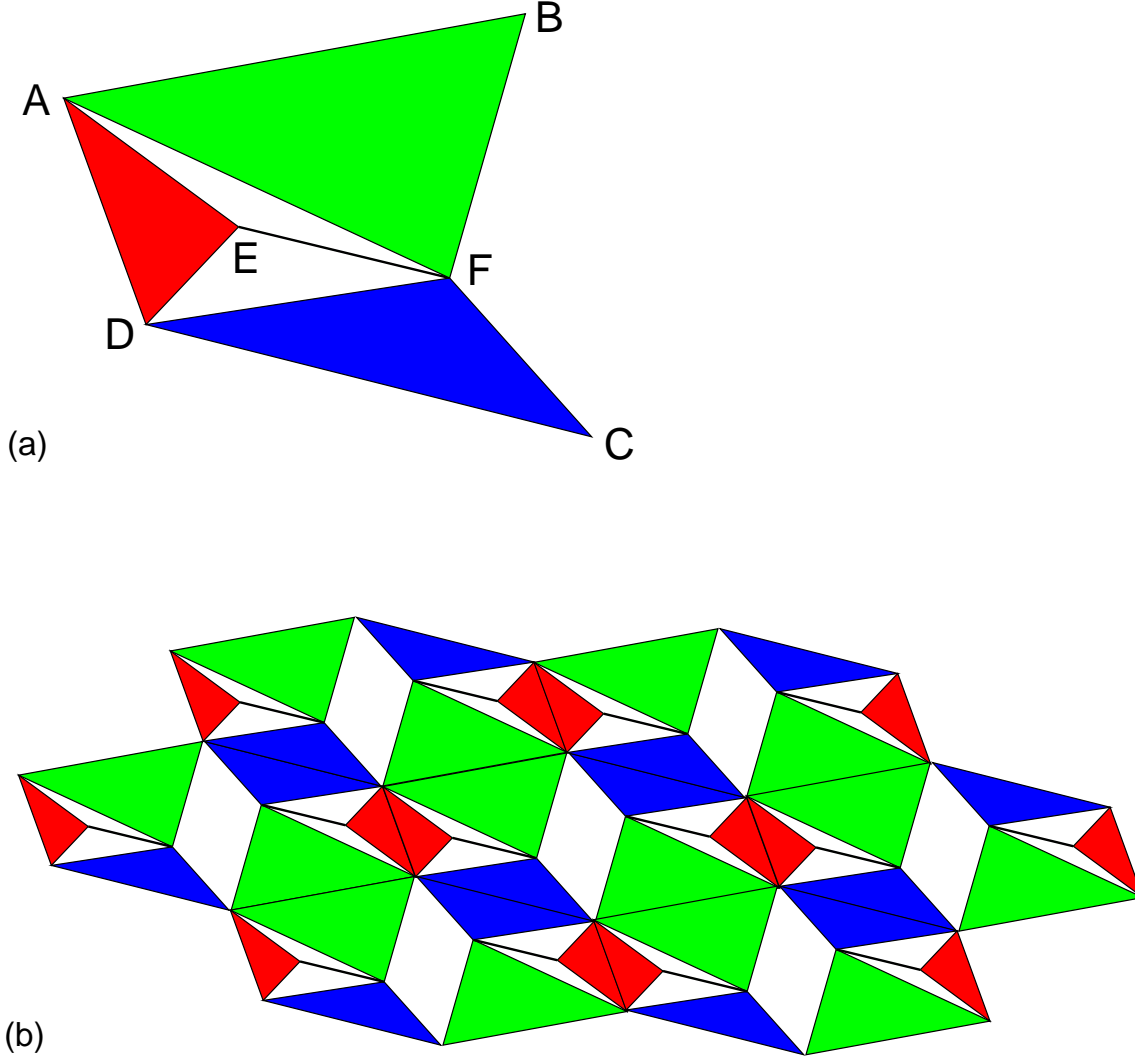


Figure 8: (a) shows a quadrilateral $ABCD$ which is a b-structure and which looks like the interior of (b) when tiled to fill all space. Here the blue, green and red triangles are u-structures, such as the one in figure 5 and EF is a rigid bar. Thus, for example, knowledge of the lengths AD and DC , determines the triangles ADE and DFC , and since the length EF is fixed, the angle ADC is determined, as is the length AF and hence the triangle ABF . Thus the quadrilateral $ABCD$ is determined (within a finite number of possibilities). Similarly if we know the lengths AB and BC , then given an angle ABC , we can determine the triangles ABF , CDF (since FC is known), and ADE (since AD is known). That angle ABC has to be chosen so EF matches the length of the rigid bar, and then the quadrilateral $ABCD$ is determined.

transmitted to the entire structure from just two actuators, so again a sparse array would be better.

4 On the route towards trimode transformer materials

Ideally one would like to construct trimode materials whose deformation is controlled by actuators in just a few cells. It is not clear if they exist or not, but one route towards their realization might be the construction of what we call a t-structure (t for trimode). It is a hexagon $ABCDEF$ with opposite sides being parallel and of the same length as shown in figure 10(a). The sides are not rigid, and there are three internal degrees of freedom. The triple consisting of the lengths of any pair of adjoining sides, such as FA and AB , plus the included angle can range anywhere inside some open connected region of three-dimensional space (i.e. the triple is not confined to lie on a surface) and knowledge of the triplet completely specifies the hexagon up to a finite number of possibilities. At this time we do not know of a mechanism of bars and pivots which realizes t-structures, nor if such a mechanism even exists.

Assuming t-structures can be made, we could consider a tiling of space with them. Now consider a cluster of two adjoining tiles, as in 10(b), and suppose there are six actuators, three in each cell, slaved so that tile 2 is always a copy of tile 1, even as tile 1 deforms. Since the lengths GC and CD and the included angle GCD are specified the adjoining tile 3 can only deform in synchrony with tiles 1 and tiles 2. By induction the same argument applies to any cell in the tiling: the deformation of the entire structure is determined by the three actuators in cell 1, and their three slave actuators in cell 2.

Now consider what we call a trimode transformer material, a tiling of t-structures where actuator cells and their adjoining slave cells are sparsely populated throughout the material, as illustrated in figure 11. Also suppose the bars inside the t-structures are not completely rigid, but have some small flexibility. Then the deformation in the neighborhood of an actuator cell should be controlled by its three actuators. So if the settings of the actuators vary slowly from one actuator cell to its nearest neighbor actuator cells, then it seems likely (assuming there is no buckling) that we could control (within limits) the macroscopic inhomogeneous deformation of the material. This may be very useful for practical applications. Of course one could do this just with a triangular array of actuator bars, but the goal is to reduce the number of actuators in the construction.

5 Open problems

There are many open problems remaining. Within the class of non-affine metamaterials using rigid bars and pivots it would be interesting to know what macroscopic deformations (besides the affine ones) can be realized by some structure. Although we have provided examples of a large class of bimode materials, it is doubtful that this class is rich enough to capture all types of affine macroscopic deformations that bimode materials can exhibit. It would be wonderful to have a characterization of possible macroscopic deformations analogous to that for affine unimode materials (Milton 2012). As mentioned,

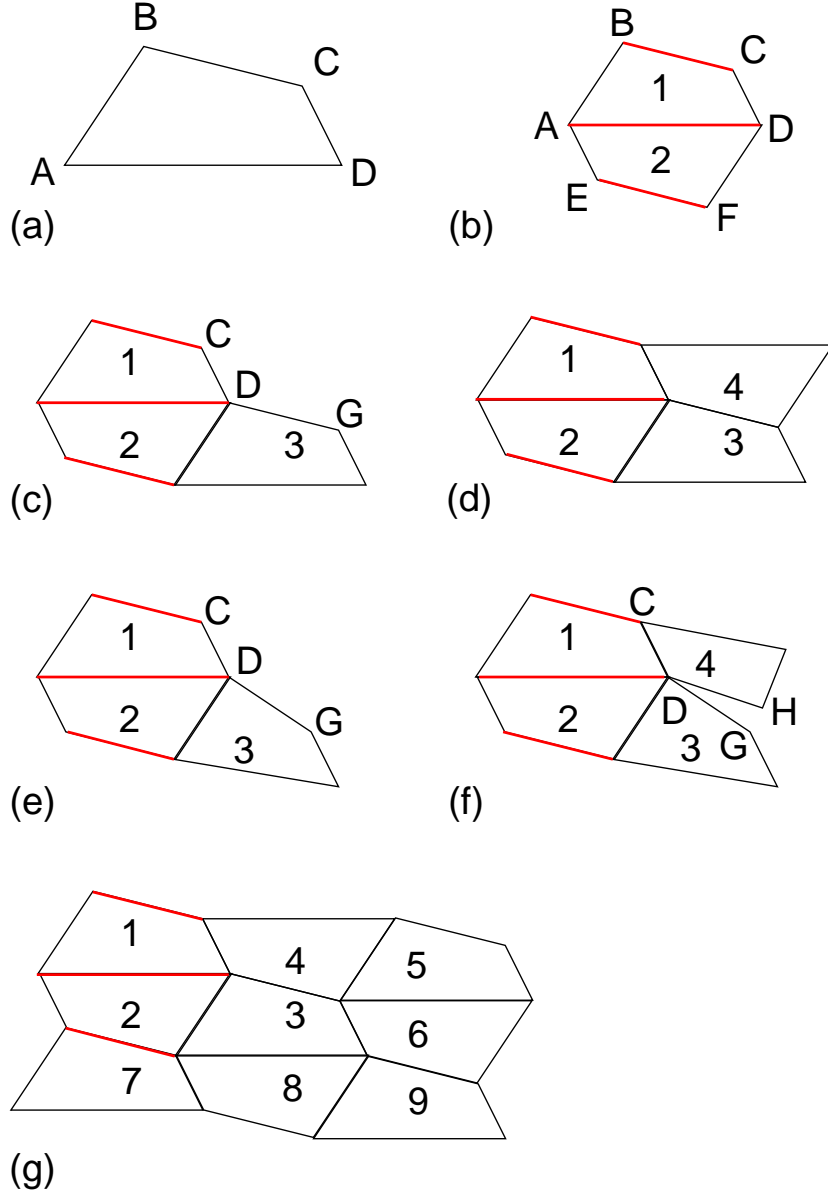


Figure 9: Sketches showing that the deformation of a tiling of b-structures is determined by the deformation of one tile. Actuators are positioned along the red lines, and finally it is argued that the actuator along the line EF is not needed. See the text for more explanation.

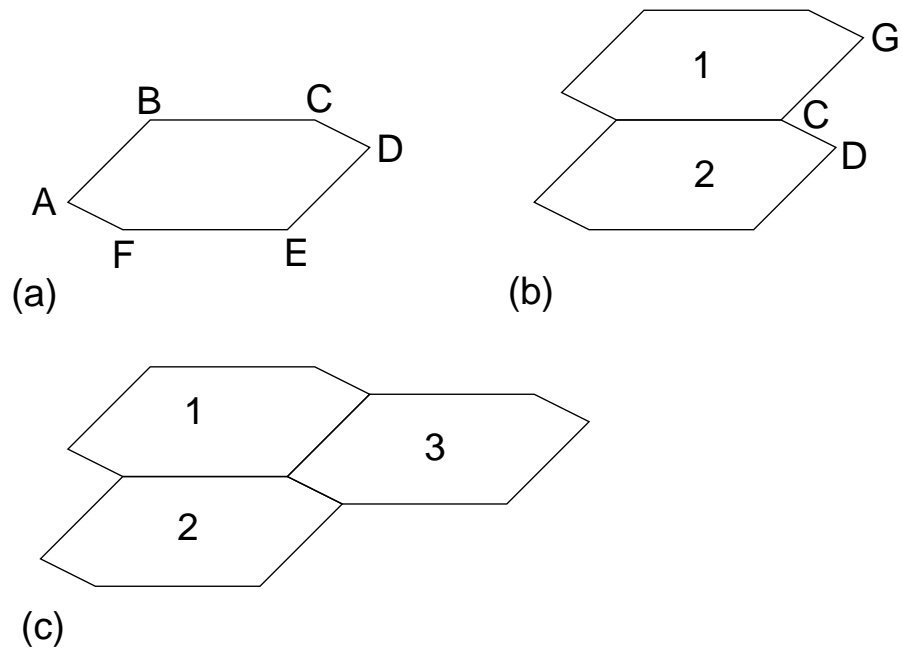


Figure 10: Sketches showing that the deformation of a tiling of t-structures is determined by the deformation of tile 1 assuming tile 2 is slaved to it. See the text for more explanation.

it is an open question as to whether there exist t-structures as described in the previous section. Even if t-structures do not exist, there may be other, yet undiscovered, routes towards realizing trimode transformer materials. Finally the territory of what affine and non-affine macroscopic deformations are possible in three dimensional bimode, trimode, quadramode, pentamode and hexamode structures remains largely unexplored: we are even lacking examples of non-linear quadramode and pentmode materials.

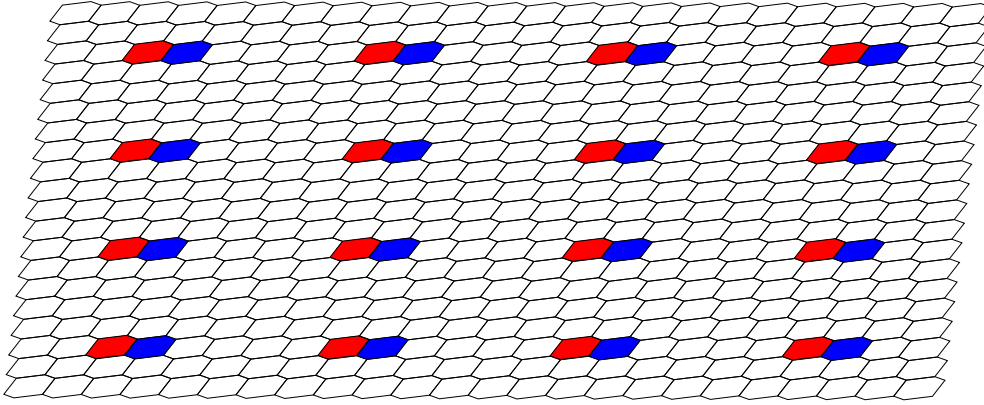


Figure 11: A tiling of t-structures containing a sparse array of actuator cells (red) and their adjoining slave cells (blue).

Acknowledgements

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